EXAMPLE 7 □
(a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
(b) Estimate the error involved in this approximation.

SOLUTION
(a) If $n = 10$, then $\Delta x = 0.1$ and Simpson's Rule gives

□ Figure 9 illustrates the calculation in Example 7. Notice that the parabolic arcs are so close to the graph of $y = e^{x^2}$ that they are practically indistinguishable from it.

$$\int_0^1 e^{x^2}dx \approx \frac{\Delta x}{3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)]$$
$$= \frac{0.1}{3}[e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36}$$
$$+ 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1]$$
$$\approx 1.462681$$

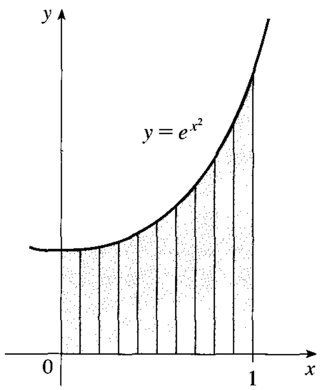

FIGURE 9

(b) The fourth derivative of $f(x) = e^{x^2}$ is

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

and so, since $0 \leqslant x \leqslant 1$, we have

$$0 \leqslant f^{(4)}(x) \leqslant (12 + 48 + 16)e^1 = 76e$$

Therefore, putting $K = 76e$, $a = 0$, $b = 1$, and $n = 10$ in (4), we see that the error is at most

$$\frac{76e(1)^5}{180(10)^4} \approx 0.000115$$

(Compare this with Example 3.) Thus, correct to three decimal places, we have

$$\int_0^1 e^{x^2}dx \approx 1.463$$

□

7.7 Exercises

1 Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is

2 The left, right, Trapezoidal, and Midpoint Rule approximations

3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin(x^2/2)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$. [illegible]

22. (a) Find the approximations T_8 and M_8 for $\int_0^1 \cos(x^2)\,dx$.
(b) Estimate the errors involved in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.00001?

23. [illegible]